JOURNAL OF COMBINATORIAL THEORY, Series A 32, 106–109 (1982)

Note

A Note on the Triangle Conjecture

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Communicated by the Managing Editors

Received March 23, 1981

We prove some particular cases of the following conjecture of Perrin and Schützenberger, known as "the triangle conjecture." Let $A = \{a, b\}$ be a two-letter alphabet, d a positive integer and let $B_d = \{a^iba^j \mid 0 \leqslant i + j \leqslant d\}$. If $X \subset B_d$ is a code, then $|X| \leqslant d + 1$.

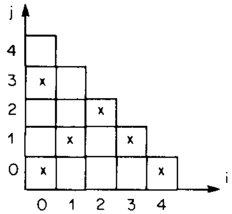
In a recent paper [1], Perrin and Schützenberger proposed a combinatorial conjecture, which was originally stated in terms of coding theory (see [2, 3]). Let $A = \{a, b\}$ be a two-letter alphabet and let A^* be the free monoid generated by A. Recall that a subset C of A^* is a code whenever the submonoid of A^* generated by C is free with base C, i.e., if the relation $c_1 \cdots c_p = c'_1 \cdots c'_q$, where $c_1,..., c_p$, $c'_1,..., c'_q$ are elements of C implies $p = q$ and $c_i = c'_i$ for $1 \leqslant i \leqslant p$. Set, for any $d > 0$, $B_d = \{a^iba^j \mid 0 \leqslant i + j \leqslant d\}$. We can now state:

THE TRIANGLE CONJECTURE. *Let $d > 0$ and $X \subset B_d$ be a set of words. If X is a code, then $|X| \leqslant d + 1$.*

In this note we shall prove some particular cases of this conjecture. The term "the triangle conjecture" originates in the following construction: if one represents every word of the form a^iba^j by a point $(i, j) \in \mathbb{N}^2$, the set B_d is

0097-3165/82/010106-04$02.00/0

represented by the triangle $\{(i, j) \in \mathbb{N}^2 \mid 0 \leqslant i + j \leqslant d\}$. For example, the set $X = \{b, ba^3, aba, a^2ba^2, a^3ba, a^4b\} \subset B_4$ can be represented by the following diagram



We shall refer to this representation to describe some properties of the set X. For example, "X has at most two columns occupied" means that there exist two integers $0 \leqslant i_1 < i_2$ such that X is contained in $a^{i_1}ba^* \cup a^{i_2}ba^*$.

In the sequel we shall need another formulation of the triangle conjecture. This new statement may appear to be somewhat involved but it is actually better adapted to our purpose. Let us first introduce a convenient definition:

A set $X \subset B_d$ satisfies property P_n whenever there exists a sequence $r_1, r_2, \dots, r_m$ of non-negative integers such that

$$|\{(x_1, \dots, x_m) \in X^m \mid x_1 \cdots x_m \in a^{r_1}ba^{r_2}b \cdots a^{r_m}ba^*\}| \geqslant n + 1.$$

The relationship with the triangle conjecture is provided by the following result:

THEOREM 1. *Let $d > 0$ be an integer, and let X be a subset of B_d which occupies exactly k rows. The following statements are equivalent:*

(a) *X is a non-code,*

(b) *X satisfies P_n for all $n > 0$,*

(c) *X satisfies P_k.*

Assume that (a) is satisfied. Then there exist $x_1, \dots, x_p, x'_1, \dots, x'_q \in X$ such that $x_1 \cdots x_p = x'_1 \cdots x'_q = u$ and $x_1 \neq x'_1$. Arguing on the number of b in u we get $p = q$. Thus u has (at least) 2 decompositions over X and therefore u^n has 2^n decompositions over X. Setting $u^n = a^{r_1}ba^{r_2}b \cdots a^{r_{np}}ba^s$, we have

$$|\{(x_1, \dots, x_{np}) \in X^{np} \mid x_1 \cdots x_{np} \in a^{r_1}b \cdots a^{r_{np}}ba^*\}| \geqslant 2^n.$$

Thus X satisfies P_{2^n-1} for all $n > 0$. Since P_n implies P_m for all $m < n$, statement (b) follows easily.

Clearly (b) implies (c). Finally, assume that (c) is satisfied. Since X satisfies P_k, there exists a sequence $r_1, \dots, r_m$ such that the set

$$E = \{(x_1, \dots, x_m) \in X^m \mid x_1 \cdots x_m \in a^{r_1}b \cdots a^{r_m}ba^*\}$$

has at least $(k+1)$ elements. Define $\varphi: E \to A^*$ by $(x_1,\dots,x_m)\varphi = x_1 \cdots x_m$. Then $|E\varphi| \leqslant k$ since X has exactly k rows occupied. Therefore there exist two distinct sequences $(x_1,\dots,x_m)$, $(x'_1,\dots,x'_m) \in X^m$ such that $x_1 \cdots x_m = x'_1 \cdots x'_m$ and X is a non-code. This concludes the proof. ∎

COROLLARY. *The triangle conjecture is equivalent with the following statement: every set $X \subset B_d$ such that $|X| > d+1$ satisfies P_n for all $n > 0$.*

As a first step towards the triangle conjecture, we prove

THEOREM 2. *Every set $X \subset B_d$ such that $|X| > d+1$ satisfies P_1 and P_2.*

Set, for $0 \leqslant k \leqslant d$, $J_k = \{i \in \{0,\dots,d\} \mid X \cap a^iba^* \mid = k\}$. Roughly speaking, J_k is the set of columns of X containing exactly k points. Assume at first that J_k is non-empty for some $k \geqslant 3$. Then there exist $0 \leqslant i \leqslant d$ and $0 \leqslant j_1 < j_2 < j_3 \leqslant d$ such that $a^iba^{j_1}$, $a^iba^{j_2}$, $a^iba^{j_3} \in X$. Setting $r_1 = i$, we have $a^iba^{j_1}$, $a^iba^{j_2}$, $a^iba^{j_3} \in a^{r_1}ba^*$ and thus X satisfies P_2 (and therefore P_1).

Assume to the contrary that J_k is empty for all $k \geqslant 3$. It follows that $|X| = |J_1| + 2|J_2| > d+1$ and J_2 is non-empty since $|J_1| \leqslant d+1$. By definition, if $i \in J_2$, X contains exactly two words $a^iba^{k_i}$, $a^iba^{k'_i}$ (with $k_i < k'_i$) having a^ib as a prefix. Select such an $i \in J_2$ with the difference $r = k'_i - k_i$ minimal. We claim that the set $(r + J_2) \cap (J_1 \cup J_2)$ is non-empty. First, we note that $r + J_2$ is contained in $\{0,\dots,d\}$ since for every $j \in J_2$, we have $r \leqslant k'_j - k_j$, and thus $j + r \leqslant j + k'_j - k_j \leqslant j + k'_j \leqslant d$. On the other hand, $J_1 \cup J_2$ is also contained in $\{0,\dots,d\}$. Now the claim holds since

$$|r + J_2| + |J_1 \cup J_2| = |J_1| + 2|J_2| > d+1.$$

Consequently, there exist $j_1 \in J_1 \cup J_2$ and $j_2 \in J_2$ such that $r = k_i - k'_i = j_1 - j_2$ hence $k_i + j_2 = k'_i + j_1$. Let t_1, t_2, t'_2 such that $a^{j_1}ba^{t_1}$, $a^{j_2}ba^{t_2}$, $a^{j_2}ba^{t'_2} \in X$. Setting $r_1 = i$ and $r_2 = k_i + j_2 = k'_i + j_1$. We have $a^iba^{k_i}a^{j_2}ba^{t_2}$, $a^iba^{k_i}a^{j_2}ba^{t'_2}$, $a^iba^{k'_i}a^{j_1}ba^{t_1} \in a^{r_1}ba^{r_2}ba^*$ and thus X satisfies P_2 and P_1. ∎

COROLLARY . *Let $X \subset B_d$ be a set having at most two rows occupied. If $|X| > d+1$, then X is a non-code.*

By Theorem 2, X satisfies P_2. Hence, by Theorem 1(c), X is a non-code. ∎

Of course, the same result holds if we replace "row" by "column." Our last result proves the triangle conjecture in an other particular case.

THEOREM 3. *Let $X \subset B_d$ be a set of words. Assume there is exactly one column of X with 2 points or more. Then if $|X| > d+1$, X is a non-code.*

By hypothesis, there exist $0 \leqslant i \leqslant d$, $r \geqslant 1$ and

$$0 \leqslant j_1 < j_2 < \cdots < j_{r+1} \leqslant d-i$$
$$\text{such that } a^i ba^{j_1}, a^i ba^{j_2}, \dots, a^i ba^{j_{r+1}} \in X. \tag{1}$$

Moreover, since $|X| > d+1$, X has at most $(r-1)$ columns unoccupied. Assume that X is a code. We shall show that X satisfies P_n for all n by proving the following stronger result: There exists a sequence $(r_k)_{k>0}$ of non-negative integers such that for all $n > 0$, the set

$$E_n = \{(x_1, \dots, x_n) \in X^n \mid x_1 \cdots x_n \in a^{r_1} ba^{r_2} b \cdots a^{r_n} b\{1, a, a^2, \dots, a^{d-i}\}\}$$

satisfies $|E_n| \geqslant n + r$.

We construct the sequence by induction. For $n = 1$, set $r_1 = i$. Then by (1), we have $|E_1| \geqslant r+1$. Associate to any $x = (x_1, \dots, x_n) \in E_n$ an integer $x\alpha \leqslant d-i$ such that $x_1 \cdots x_n = a^{r_1} ba^{r_2} b \cdots a^{r_n} ba^{x\alpha}$. Since X is a code, α is one-to-one. Set $s = \max_{x \in E_n} x\alpha$ and $r_{n+1} = i + s$. We claim that $|E_{n+1}| \geqslant n + r + 1$. Indeed, let $(x_1, \dots, x_n) = s\alpha^{-1}$. Then E_{n+1} contains $(x_1, \dots, x_n, a^i ba^{j_1}), \dots, (x_1, \dots, x_n, a^i ba^{j_{r+1}})$. Moreover, since $|E_n| \geqslant n + r$ by induction, we have $|E_n \alpha| \geqslant n + r$ and since X has at most $(r-1)$ columns unoccupied, we can select n indices $s_1, \dots, s_n \in E_n \alpha \backslash \{s\}$ such that the columns of index $i + s - s_k$ $(1 \leqslant k \leqslant n)$ contain exactly one element $u_k = a^{i+s-s_k} ba^{t_k}$. Since we have, for $1 \leqslant k \leqslant n$, $t_k \leqslant d - (i + s - s_k) \leqslant d - i$, E_{n+1} contains $(s_k \alpha^{-1}, u_k)$ for $1 \leqslant k \leqslant n$ and the claim holds. Thus X satisfies P_n for all $n > 0$, in contradiction with Theorem 1.

Thus X is a non-code. ∎

References

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